A PROOF OF LEMMA 2

Proof. The proof technique is standard, and can be found in Zinkevich (2003); Hazan et al. (2016).

First, we prove the regret bound of (21). Note that by Definition 2, $s_t^{\eta}(\mathbf{x})$ is $2\eta^2 G^2$ -strongly convex. For convince, we denote $\alpha_{t+1} = 1/(2\eta^2 G^2 t)$, $\lambda^s = 2\eta^2 G^2$, and define the upper bound of the gradients of $s_t^{\eta}(\mathbf{x})$ as

$$\max_{\mathbf{x}\in\mathcal{D}}kr s_t^{\eta}(\mathbf{x})k = \max_{\mathbf{x}\in\mathcal{D}}k\eta \mathbf{g}_t + 2\eta^2 G^2(\mathbf{x} \quad \mathbf{x}_t)k \quad G\eta + 2\eta^2 G^2 D =: G^s.$$

By the update rule of $\mathbf{x}_{t+1}^{\eta,s}$, we have

$$k\mathbf{x}_{t+1}^{\eta,s} \quad \mathbf{u}k = \left\| \begin{array}{cc} I_{d} \left(\mathbf{x}_{t}^{\eta,s} & \alpha_{t+1} \upharpoonright s_{t}^{\eta}(\mathbf{x}_{t}^{\eta,s}) \right) & \mathbf{u} \right\| \\ k\mathbf{x}_{t}^{\eta,s} & \alpha_{t+1} \upharpoonright s_{t}^{\eta}(\mathbf{x}_{t}^{\eta,s}) & \mathbf{u}k \\ = k\mathbf{x}_{t}^{\eta,s} & \mathbf{u}k^{2} + \alpha_{t+1}^{2} k \upharpoonright s_{t}^{\eta}(\mathbf{x}_{t}^{\eta,s})k^{2} & 2\alpha_{t+1}(\mathbf{x}_{t}^{\eta,s} & \mathbf{u})^{\top} \upharpoonright s_{t}^{\eta}(\mathbf{x}_{t}^{\eta,s}). \end{array}$$

$$(28)$$

Hence,

$$2(\mathbf{x}_t^{\eta,s} \quad \mathbf{u})^{\top} \cap s_t^{\eta}(\mathbf{x}_t^{\eta,s}) \quad \frac{k\mathbf{x}_t^{\eta,s} \quad \mathbf{u}k \quad k\mathbf{x}_{t+1}^{\eta,s} \quad \mathbf{u}k^2}{\alpha_{t+1}} + \alpha_{t+1}(G^s)^2.$$
(29)

Summing over 1 to T and applying definition 2, we get

$$2\sum_{t=1}^{T} s_{t}^{\eta}(\mathbf{x}_{t}^{\eta,s}) = 2\sum_{t=1}^{T} s_{t}^{\eta}(\mathbf{u}) = \sum_{t=1}^{T} k \mathbf{x}_{t}^{\eta,s} = \mathbf{u}k^{2} \left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_{t}} - \lambda^{s}\right) + (G^{s})^{2} \sum_{t=1}^{T} \alpha_{t+1} = \frac{(G^{s})^{2}}{\lambda^{s}} (1 + \log T).$$
(30)

Note that $\eta = \frac{1}{5DG}$. We have

$$(G^{s})^{2} = G^{2}\eta^{2} + 4\eta^{3}G^{3}D + 4\eta^{4}G^{4}D^{2} \qquad G^{2}\eta^{2} + \frac{4\eta^{2}G^{2}}{5} + \frac{4\eta^{2}G^{2}}{25} \qquad 2\eta^{2}G^{2} = \lambda^{s}.$$
 (31)

Next, we prove the regret bound of (22). We start with the following inequality

$$\Gamma \ell_t^{\eta}(\mathbf{x}) (\Gamma \ell_t^{\eta}(\mathbf{x}))^{\top} = \eta^2 \mathbf{g}_t \mathbf{g}_t^{\top} + 4\eta^3 \mathbf{g}_t (\mathbf{x} \quad \mathbf{x}_t)^{\top} \mathbf{g}_t \mathbf{g}_t^{\top} + 4\eta^4 \mathbf{g}_t \mathbf{g}_t^{\top} (\mathbf{x} \quad \mathbf{x}_t) (\mathbf{x} \quad \mathbf{x}_t)^{\top} \mathbf{g}_t \mathbf{g}_t^{\top}$$

$$= \eta^2 \mathbf{g}_t \mathbf{g}_t^{\top} + \mathbf{g}_t \left(4\eta^3 (\mathbf{x} \quad \mathbf{x}_t)^{\top} \mathbf{g}_t + 4\eta^4 \left((\mathbf{x} \quad \mathbf{x}_t)^{\top} \mathbf{g}_t \right)^2 \right) \mathbf{g}_t^{\top}$$

$$= \eta^2 \mathbf{g}_t \mathbf{g}_t^{\top} = \Gamma^2 \ell_t^{\eta} (\mathbf{x})$$

$$(32)$$

where $\Gamma^2 \ell_t^{\eta}(\mathbf{x})$ denotes the Hessian matrix. The inequality implies that $\Gamma^2 \ell_t^{\eta}(\mathbf{x}) = \Gamma \ell_t^{\eta}(\mathbf{x}) (\Gamma \ell_t^{\eta}(\mathbf{x}))^{\top}$. According to Lemma 4.1 in Hazan et al. (2016), $\ell_t^{\eta}(\mathbf{x})$ is 1-exp-concave. Next, we prove that the gradient of $\ell_t^{\eta}(\mathbf{x})$ can be upper bounded as follows

$$\max_{\mathbf{x}\in\mathcal{D}} k \Gamma \ell_t^{\eta}(\mathbf{x}) k \quad \eta G + 2\eta^2 G^2 D \quad \frac{7}{25D} = G^{\ell}.$$
(33)

By Theorem 4.3 in Hazan et al. (2016), we have

$$\sum_{t=1}^{T} \ell_t^{\eta}(\mathbf{x}_t^{\eta,\ell}) = \sum_{t=1}^{T} \ell_t^{\eta}(\mathbf{u}) = 5(1 + G^{\ell}D)d\log T = 10d\log T.$$
(34)

Finally, we prove the regret bound of (23). Note that the gradient of $c_t(\mathbf{x})$ is upper bounded by $\max_{\mathbf{x}\in\mathcal{D}} k \cap c_t(\mathbf{x}) k \eta^c G$. Define $m_t = \frac{D}{\eta^c G \sqrt{t}}$. By the convexity of $c_t(\mathbf{x})$, we have $\partial \mathbf{u} \geq D$,

$$c_t \left(\mathbf{x}_t^c \right) \quad c_t \left(\mathbf{u} \right) \quad \left(\mathbf{x}_t^c \quad \mathbf{u} \right)^\top \land c_t \left(\mathbf{x}_t^c \right).$$
(35)

On the other hand, according to the update rule of \mathbf{x}_{t+1}^{c} , we have

$$k\mathbf{x}_{t+1}^{c} \quad \mathbf{u}k^{2} = k \frac{I_{d}}{\mathcal{D}} (\mathbf{x}_{t}^{c} \quad m_{t} \cap c_{t}(\mathbf{x}_{t}^{c})) \quad \mathbf{u}k^{2}$$

$$k\mathbf{x}_{t}^{c} \quad m_{t} \cap c_{t}(\mathbf{x}_{t}^{c}) \quad \mathbf{u}k^{2}$$

$$= k\mathbf{x}_{t}^{c} \quad \mathbf{u}k^{2} + m_{t}^{2}k \cap c_{t}(\mathbf{x}_{t}^{c})k^{2} \quad 2m_{t}(\mathbf{x}_{t}^{c} \quad \mathbf{u})^{\top} \cap c_{t}(\mathbf{x}_{t}^{c})$$
(36)

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