$g_i$

$+$

# Supplementary Material for Mixed Optimization for Smooth Functions

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Before proving the lemmas we recall the definition of $\mathcal{F}(\mathbf{w})$, $\mathcal{F}'(\mathbf{w})$, $\mathbf{g}$, and $\widehat{g}_i(\mathbf{w})$ as:

$$\begin{aligned}
\mathcal{F}(\mathbf{w}) &= \frac{\lambda}{2}\|\mathbf{w}\|^2 + \lambda\langle \mathbf{w}, \bar{\mathbf{w}}\rangle + \frac{1}{n}\sum_{i\quad 1}^{n} g_i(\mathbf{w} + \bar{\mathbf{w}}),\\
\mathcal{F}'(\mathbf{w}) &= \frac{\lambda}{2\gamma}\|\mathbf{w}\|^2 + \frac{\lambda}{\gamma}\langle \mathbf{w}, \bar{\mathbf{w}}'\rangle + \frac{1}{n}\sum_{i\quad 1}^{n} g_i(\mathbf{w} + \bar{\mathbf{w}}'),\\
\mathbf{g} &= \lambda\bar{\mathbf{w}} + \frac{1}{n}\sum_{i\quad 1}^{n} \nabla g_i(\bar{\mathbf{w}}),\\
\widehat{g}_i(\mathbf{w}) &= g_i(\mathbf{w} + \bar{\mathbf{w}}) - \langle \mathbf{w}, \nabla g_i(\bar{\mathbf{w}})\rangle.
\end{aligned}$$

We also recall that $\widehat{\mathbf{w}}_*$ and $\widehat{\mathbf{w}}'_*$ are the optimal solutions that minimize $\mathcal{F}(\mathbf{w})$ and $\mathcal{F}'(\mathbf{w})$ over the domain $\mathcal{W}_k$ and $\mathcal{W}_{k+1}$, respectively.

**Lemma 1.** $t$ $\widehat{\mathbf{w}}_*\|^2$

$$\mathcal{F}(\mathbf{w}_t) - \mathcal{F}(\widehat{\mathbf{w}}_*) \leq \|\mathbf{w}_t - \widehat{\mathbf{w}}_*\|^2 +$$

where the first inequality follows from the fact that $\mathbf{w}_{t+1}$ in the minimizer of the following optimization problem:

$$\mathbf{w}_{t+1} = \mathop{\arg\min}_{\mathbf{w}\in\mathcal{W}\wedge\|\mathbf{w}-\bar{\mathbf{w}}\|\leq\Delta} \langle \mathbf{g} + \nabla\widehat{g}_{i_t}(\mathbf{w}_t) + \lambda\mathbf{w}_t, \mathbf{w} - \mathbf{w}_t\rangle + \frac{\|\mathbf{w}-\mathbf{w}_t\|^2}{2\eta}.$$

Therefore, we obtain

$$\begin{aligned}
&\mathcal{F}(\mathbf{w}_t) - \mathcal{F}(\widehat{\mathbf{w}}_*) \\
\leq\; & \frac{\|\mathbf{w}_t - \widehat{\mathbf{w}}_*\|^2}{2\eta} - \frac{\|\mathbf{w}_{t+1} - \widehat{\mathbf{w}}_*\|^2}{2\eta} - \frac{\lambda}{2}\|\mathbf{w}_t - \widehat{\mathbf{w}}_*\|^2 \\
& + \langle \mathbf{g}, \mathbf{w}_t - \mathbf{w}_{t+1}\rangle + \frac{\eta}{2}\|\nabla\widehat{g}_{i_t}(\mathbf{w}_t) + \lambda\mathbf{w}_t\|^2 + \left\langle \nabla\widehat{\mathcal{F}}(\widehat{\mathbf{w}}_*) - \nabla\widehat{g}_{i_t}(\widehat{\mathbf{w}}_*), \mathbf{w}_t - \widehat{\mathbf{w}}_*\right\rangle \\
& + \left\langle -\nabla\widehat{g}_{i_t}(\mathbf{w}_t) + \nabla\widehat{g}_{i_t}(\widehat{\mathbf{w}}_*) - \nabla\widehat{\mathcal{F}}(\widehat{\mathbf{w}}_*) + \nabla\widehat{\mathcal{F}}(\mathbf{w}_t), \mathbf{w}_t - \widehat{\mathbf{w}}_*\right\rangle,
\end{aligned}$$

as desired. □

We now turn to prove the upper bound on $A_T$.

**Lemma 2.**

$$A_T \leq 6\beta^2\Delta^2 T$$

*Proof.* We bound $A_T$ as

$$\begin{aligned}
A_T &= \sum_{t=1}^{T} \|\nabla\widehat{g}_{i_t}(\mathbf{w}_t) + \lambda\mathbf{w}_t\|^2 \\
&\leq \sum_{t=1}^{T} 2\|\nabla\widehat{g}_{i_t}(\mathbf{w}_t)\|^2 + 2\lambda^2\|\mathbf{w}_t\|^2 \\
&\leq \sum_{t=1}^{T} 2\lambda^2\Delta^2 + 2\|\nabla\widehat{g}_{i_t}(\mathbf{w}_t) - \nabla\widehat{g}_{i_t}(\widehat{\mathbf{w}}_*) + \nabla\widehat{g}_{i_t}(\widehat{\mathbf{w}}_*)\|^2 \leq 6\beta^2\Delta^2 T
\end{aligned}$$

where the second inequality follows $(a+b)^2 \leq 2(a^2+b^2)$ and the last inequality follows from the smoothness assumption. □

**Lemma 3.** *With a probability* $1-2\delta$*, we have*

$$B_T \leq \beta\Delta^2\left(\ln\frac{1}{\delta} + \sqrt{2T\ln\frac{1}{\delta}}\right) \quad \textit{and} \quad C_T \leq 2\beta\Delta^2\left(\ln\frac{1}{\delta} + \sqrt{2T\ln\frac{1}{\delta}}\right)$$

The proof is based on the Berstein inequality for Martingales [1] which is restated here for completeness.

**Theorem 1.** *(Bernstein's inequality for martingales). Let* $X_1, \ldots, X_n$ *be a bounded martingale difference sequence with respect to the filtration* $\mathcal{F} = (\mathcal{F}_i)_{1\leq i\leq n}$ *and with* $\|X_i\| \leq K$*. Let*

$$S_i = \sum_{j=1}^{i} X_j$$

*be the associated martingale. Denote the sum of the conditional variances by*

$$\Sigma_n^2 = \sum_{t=1}^{n} \mathbb{E}\left[X_t^2|\mathcal{F}_{t-1}\right],$$

*Then for all constants* $t$*,* $\nu > 0$*,*

$$\Pr\left[\max_{i=1,\ldots,n} S_i > t \textit{ and } \Sigma_n^2 \leq \nu\right] \leq \exp\left(-\frac{t^2}{2(\nu + Kt/3)}\right),$$

*and therefore,*

$$\Pr\left[\max_{i=1,\ldots,n} S_i > \sqrt{2\nu t} + \frac{\sqrt{2}}{3}Kt \textit{ and } \Sigma_n^2 \leq \nu\right] \leq e^{-t}.$$

Equipped with this theorem, we are now in a position to upper bound $B_T$ and $C_T$ as follows.

*Proof.* (of Lemma 3) Denote $X_t = \langle \nabla \widehat{g}_{i_t}(\widehat{\mathbf{w}}_*) - \nabla \widehat{\mathcal{F}}(\widehat{\mathbf{w}}_*), \mathbf{w}_t - \widehat{\mathbf{w}}_* \rangle$. We have that the conditional expectation of $X_t$, given randomness in previous rounds, is $\mathbb{E}_{t-1}[X_t] = 0$. We now apply Theorem 1 to the sum of martingale differences. In particular, we have, with a probability $1 - e^{-t}$,

$$B_T \leq \frac{\sqrt{2}}{3} K t + \sqrt{2 \Sigma t}$$

where

$$\begin{aligned} K &= \max_{1 \leq t \leq T} \langle \nabla \widehat{g}_{i_t}(\widehat{\mathbf{w}}_*) - \nabla \widehat{\mathcal{F}}(\widehat{\mathbf{w}}_*), \mathbf{w}_t - \widehat{\mathbf{w}}_* \rangle \leq 2\beta\Delta^2 \\ \Sigma &= \sum_{t \quad 1}^{T} \mathbb{E}_t \left[ |\langle \nabla \widehat{g}_{i_t}(\widehat{\mathbf{w}}_*) - \nabla \widehat{\mathcal{F}}(\widehat{\mathbf{w}}_*), \mathbf{w}_t - \widehat{\mathbf{w}}_* \rangle|^2 \right] \leq \beta^2 \Delta \ \ T \end{aligned}$$

Hence, with a probability $1 - \delta$, we have

$$B_T \leq \beta\Delta^2 \left( \ln \frac{1}{\delta} + \sqrt{2T \ln \frac{1}{\delta}} \right)$$

Similar, for $C_T$, we have, with a probability $1 - \delta$,

$$C_T \leq 2\beta\Delta^2 \left( \ln \frac{1}{\delta} + \sqrt{2T \ln \frac{1}{\delta}} \right)$$

□

**Lemma 4.** $\|\widehat{\mathbf{w}}'_*\| \leq \gamma \|\widetilde{\mathbf{w}} - \widehat{\mathbf{w}}_*\|$.

*Proof.* We rewrite $\mathcal{F}(\mathbf{w})$ as

$$\begin{aligned} \mathcal{F}(\mathbf{w}) &= \frac{\lambda}{2}\|\mathbf{w}\|^2 + \lambda\langle \mathbf{w}, \bar{\mathbf{w}} \rangle + \frac{1}{n}\sum_{i \quad 1}^{n} g_i(\mathbf{w} + \bar{\mathbf{w}}) \\ &= \frac{\lambda}{2}\|\mathbf{w} - \widetilde{\mathbf{w}} + \widetilde{\mathbf{w}}\|^2 + \lambda\langle \mathbf{w} - \widetilde{\mathbf{w}} + \widetilde{\mathbf{w}}, \bar{\mathbf{w}} \rangle + \frac{1}{n}\sum_{i \quad 1}^{n} g_i(\mathbf{w} - \widetilde{\mathbf{w}} + \bar{\mathbf{w}}') \end{aligned}$$

Define $\mathbf{z} = \mathbf{w} - \widetilde{\mathbf{w}}$. We have

$$\begin{aligned} \mathcal{F}(\mathbf{w}) &= \frac{\lambda}{2}\|\mathbf{z} + \widetilde{\mathbf{w}}\|^2 + \lambda\langle \mathbf{z}, \bar{\mathbf{w}} \rangle + \lambda\langle \widetilde{\mathbf{w}}, \bar{\mathbf{w}} \rangle + \frac{1}{n}\sum_{i \quad 1}^{n} g_i(\mathbf{z} + \bar{\mathbf{w}}') \\ &= \frac{\lambda}{2}\|\mathbf{z}\|^2 + \lambda\langle \mathbf{z}, \bar{\mathbf{w}}' \rangle + \frac{1}{n}\sum_{i \quad 1}^{n} g_i(\mathbf{z} + \bar{\mathbf{w}}') + \frac{\lambda}{2}\|\widetilde{\mathbf{w}}\|^2 + \lambda\langle \widetilde{\mathbf{w}}, \bar{\mathbf{w}} \rangle \\ &= \widetilde{\mathcal{F}}(\mathbf{z}) + \frac{\lambda}{2}\|\widetilde{\mathbf{w}}\|^2 + \lambda\langle \widetilde{\mathbf{w}}, \bar{\mathbf{w}} \rangle \end{aligned}$$

where

$$\widetilde{\mathcal{F}}(\mathbf{z}) = \frac{\lambda}{2}\|\mathbf{z}\|^2 + \lambda\langle \mathbf{z}, \bar{\mathbf{w}}' \rangle + \frac{1}{n}\sum_{i \quad 1}^{n} g_i(\mathbf{z} + \bar{\mathbf{w}}')$$

Define $\widetilde{\mathbf{w}}_* = \widehat{\mathbf{w}}_* - \widetilde{\mathbf{w}}$. Evidently, $\widetilde{\mathbf{w}}_*$ minimizes $\widetilde{\mathcal{F}}(\mathbf{w})$. The only difference between $\widetilde{\mathcal{F}}(\mathbf{w})$ and $F'(\mathbf{w})$ is that they use different modulus of strong convexity $\lambda$. Thus, following [2], we have

$$\|\widetilde{\mathbf{w}}_* - \widehat{\mathbf{w}}'_*\| \leq \frac{1 - \gamma^{-1}}{\gamma^{-1}}\|\widetilde{\mathbf{w}}_*\| \leq (\gamma - 1)\|\widetilde{\mathbf{w}}_*\|$$

Hence,

$$\|\widehat{\mathbf{w}}'_*\| \leq \gamma\|\widetilde{\mathbf{w}}_*\| = \gamma\|\widehat{\mathbf{w}}_* - \widetilde{\mathbf{w}}\|$$

which completes the proofs. □

## References

[1] S. Boucheron, G. Lugosi, and O. Bousquet. Concentration inequalities. In *Advanced Lectures on Machine Learning*, pages 208–240, 2003.

[2] L. Zhang, M. Mahdavi, R. Jin, T. Yang, and S. Zhu. Recovering the optimal solution by dual random projection. In *COLT*, 2013.